Wave propagation in randomly-layered media

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1. J.-P. Fouque et al. *Wave propagation and time reversal in randomly layered media*. T. 56. Stochastic Modelling and Applied Probability. Springer, 2007

Outline

Outline

Acoustic wave propagation in a homogeneous medium

The pressure $p(t,z)$ and velocity $\boldsymbol{u}(t,z)$ in a homogeneous medium with density ρ_0 and bulk modulus K_0 verify

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \boldsymbol{F},$$

$$\frac{1}{K_0}\frac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} = 0.$$

where $\boldsymbol{F}$ is an external force.

This can be rewritten as a standard wave equation for the pressure only :

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = -\nabla \cdot \boldsymbol{F}$$

where $c_0 = \sqrt{K_0/\rho_0}$ is the wave velocity in the medium.

Decomposition into right- and left-going modes

For a large part of this talk, we concentrate on the 1D case.

Acoustic wave equation in a homogeneous 1D medium

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = F,$$

$$\frac{1}{K_0}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0,$$

The system can be rewritten in terms of right-going wave $A(t,z)$ and left-going waves $B(t,z)$ as

$$\frac{\partial A}{\partial z} + \frac{1}{c_0}\frac{\partial A}{\partial t} = \delta(z)f(t)$$

$$\frac{\partial B}{\partial z} - \frac{1}{c_0}\frac{\partial B}{\partial t} = -\delta(z)f(t)$$

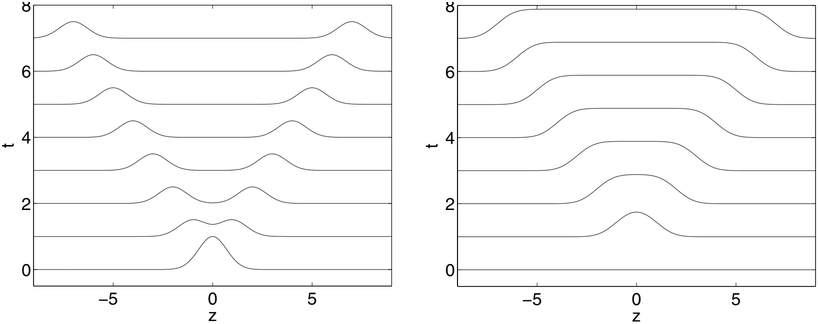
where $A(t,z) = \zeta_0^{-1/2}p(t,z) + \zeta_0^{1/2}u(t,z)$ and $B(t,z) = -\zeta_0^{-1/2}p(t,z) + \zeta_0^{1/2}u(t,z)$ and $\zeta_0 = \sqrt{K_0\rho_0}$ is the impedance. We have assumed $F(t,z) = 2\zeta_0^{1/2}\delta(z)f(t)$.

- The modes propagate independently one from the other (only in a homogeneous medium) away from the source.

$$A(t,z) = a\left(t - \frac{z}{c_0}\right), \quad B(t,z) = b\left(t + \frac{z}{c_0}\right)$$

- The point load generates two waves (one in each direction) of equal energy

Numerical illustration of acoustic wave propagation in a 1D medium



(a) $p_0(z) = \exp(-z^2), \quad p_1(z) = 0$

(b) $p_0(z) = 0, \quad p_1(z) = \exp(-z^2)$

FIGURE – Waves generated for two sets of initial conditions $p(t = 0, z) = p_0(z)$ and $\partial_t p(t = 0, z) = p_1(z)$ and no load. The spatial profiles of the field $p(t, z)$ are plotted at different times.

Numerical illustration of acoustic wave propagation in a 1D medium

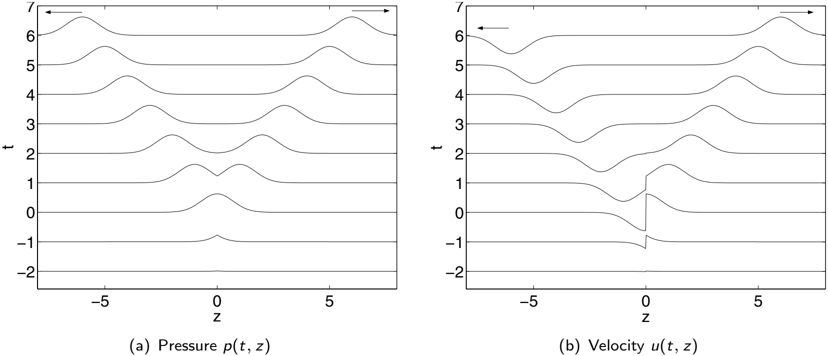
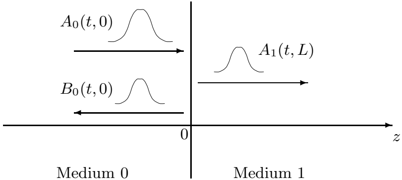


FIGURE – Waves generated by a point load $F(z, t) = \delta(z) \exp(-t^2)$ in a medium initially at rest.

Outline

Scattering by a single interface



We consider two half-spaces, separated by an interface at $z = 0$

$$\rho(z) = \begin{cases} \rho_0 & \text{if } z < 0 \\ \rho_1 & \text{if } z > 0 \end{cases}, \quad K(z) = \begin{cases} K_0 & \text{if } z < 0 \\ K_1 & \text{if } z > 0 \end{cases}, \quad c_j = \sqrt{\frac{K_j}{\rho_j}}, \quad \zeta_j = \sqrt{K_j \rho_j}.$$

In terms of right- and left-going modes in each of the half-spaces, the equations are

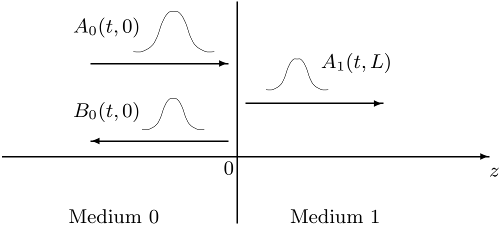
$$\frac{\partial}{\partial z} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{c_0} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \quad \frac{\partial}{\partial z} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \frac{1}{c_1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}.$$

The continuity of pressure and velocity at the interface $z = 0$ induces in terms of modes

$$\begin{bmatrix} A_1(t,0) \\ B_1(t,0) \end{bmatrix} = \mathbf{J}_0 \begin{bmatrix} A_0(t,0) \\ B_0(t,0) \end{bmatrix}, \quad \mathbf{J}_0 = \begin{bmatrix} r_0^{(+)} & r_0^{(-)} \\ r_0^{(-)} & r_0^{(+)} \end{bmatrix},$$

where $r_0^{(\pm)} = \left(\sqrt{\zeta_1/\zeta_0} \pm \sqrt{\zeta_0/\zeta_1} \right)/2$ and $\mathbf{J}_0$ is the interface propagator.

Reflection and transmission at a single interface



Assuming an incoming wave from the left on the interface, and no incoming wave from the right

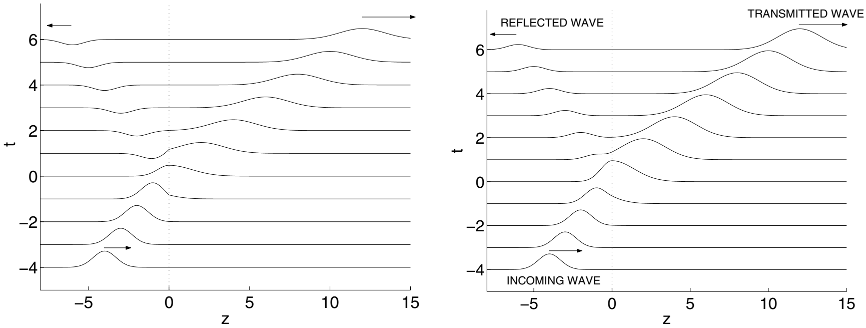
$$A_0(t,0) = f(t), \quad B_1(t,0) = 0,$$

the reflection $\mathcal{R}_0$ and transmission $\mathcal{T}_0$ coefficients at the interface are defined as

$$B_0(t,0) = \mathcal{R}_0 f(t) = -\frac{r_0^{(-)}}{r_0^{(+)}} f(t), \quad A_1(t,0) = \mathcal{T}_0 f(t) = \frac{1}{r_0^{(-)}} f(t),$$

with $\mathcal{R}_0^2 + \mathcal{T}_0^2 = 1$.

Numerical illustration of reflection and transmission at a single interface

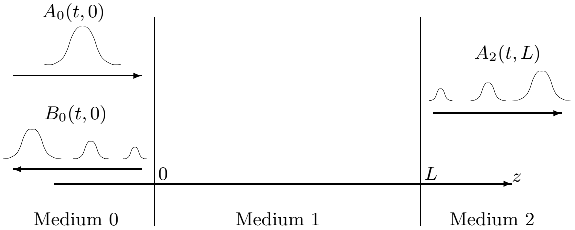


(a) Pressure $p(t, z)$ (b) Velocity $u(t, z)$

FIGURE – Waves generated at a single interface by a wave incoming from the left.

Outline

Scattering by a homogeneous slab



We consider a homogeneous slab of thickness L in-between two homogeneous half-spaces

$$\rho(z) = \begin{cases} \rho_0 & \text{if } z < 0 \\ \rho_1 & \text{if } z \in [0, L] \\ \rho_2 & \text{if } z > L \end{cases}, \quad K(z) = \begin{cases} K_0 & \text{if } z < 0 \\ K_1 & \text{if } z \in [0, L] \\ K_2 & \text{if } z > L \end{cases}, \quad c_j = \sqrt{\frac{K_j}{\rho_j}}, \quad \zeta_j = \sqrt{K_j \rho_j}.$$

In terms of right- and left-going modes in each of the domains, $0 \leq j \leq 2$, the equations are

$$\frac{\partial}{\partial z} \begin{bmatrix} A_j \\ B_j \end{bmatrix} = \frac{1}{c_j} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A_j \\ B_j \end{bmatrix}.$$

We assume, as earlier,

$$A_0(t,0) = f(t), \quad B_2(t,L) = 0$$

Frequency-dependent interface propagator

The continuity of pressure and velocity at the interfaces $z = 0$ and $z = L$ induces

$$\begin{bmatrix} A_1(t,0) \\ B_1(t,0) \end{bmatrix} = \mathbf{J}_0 \begin{bmatrix} f(t) \\ B_0(t,0) \end{bmatrix}, \quad \begin{bmatrix} A_2(t,L) \\ 0 \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} A_1(t,L) \\ B_1(t,L) \end{bmatrix}, \quad \text{with } \mathbf{J}_j = \begin{bmatrix} r_j^{(+)} & r_j^{(-)} \\ r_j^{(-)} & r_j^{(+)} \end{bmatrix},$$

and $r_j^{(\pm)} = \left(\sqrt{\zeta_{j+1}/\zeta_j} \pm \sqrt{\zeta_j/\zeta_{j+1}}\right)/2$. Equivalently :

$$\begin{bmatrix} a_1(t,0) \\ b_1(t,0) \end{bmatrix} = \mathbf{J}_0 \begin{bmatrix} f(t) \\ b_0(t) \end{bmatrix}, \quad \begin{bmatrix} a_2(t) \\ 0 \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} a_1(t - L/c_1) \\ b_1(t + L/c_1) \end{bmatrix}.$$

The time delay makes the situation more complex to handle than for a single interface. In the frequency domain, time shifts are transformed into phase factors, as

$$\begin{bmatrix} \hat{a}_1(\omega) \\ \hat{b}_1(\omega) \end{bmatrix} = \mathbf{J}_0 \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(\omega) \end{bmatrix}, \quad \begin{bmatrix} \hat{a}_2(\omega) \\ 0 \end{bmatrix} = \hat{\mathbf{J}}_1(\omega) \begin{bmatrix} \hat{a}_1(\omega) \\ \hat{b}_1(\omega) \end{bmatrix}, \quad \hat{\mathbf{J}}_1(\omega) = \begin{bmatrix} r_1^{(+)} e^{\frac{i\omega L}{c_1}} & r_1^{(-)} e^{-\frac{i\omega L}{c_1}} \\ r_1^{(-)} e^{\frac{i\omega L}{c_1}} & r_1^{(+)} e^{-\frac{i\omega L}{c_1}} \end{bmatrix},$$

where a hat denotes a Fourier transform $\hat{a}_j(\omega) = \int_{\mathbb{R}} a_j(t) e^{i\omega t} dt$ and

$$\int_{\mathbb{R}} a_j(t - t_0) e^{i\omega t} dt = \hat{a}_j(\omega) e^{i\omega t_0}$$

Reflection and transmission of a homogeneous slab

Finally

$$\begin{bmatrix} \hat{a}_2(\omega) \\ 0 \end{bmatrix} = \hat{\mathbf{K}}_0(\omega) \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(\omega) \end{bmatrix}, \quad \hat{\mathbf{K}}_0(\omega) = \hat{\mathbf{J}}_1(\omega)\mathbf{J}_0 = \begin{bmatrix} \hat{U}(\omega) & \overline{\hat{V}(\omega)} \\ \hat{V}(\omega) & \overline{\hat{U}(\omega)} \end{bmatrix},$$

Defining the reflection $\hat{\mathcal{R}}(\omega)$ and transmission $\hat{\mathcal{T}}(\omega)$ coefficients of the slab as

$$\hat{b}_0(\omega) = \hat{\mathcal{R}}(\omega)\hat{f}(\omega), \quad \hat{a}_2(\omega) = \hat{\mathcal{T}}(\omega)\hat{f}(\omega),$$

we obtain

$$\hat{\mathcal{R}}(\omega) = -\frac{\hat{V}(\omega)}{\overline{\hat{U}(\omega)}} = \frac{\mathcal{R}_1 e^{2i\frac{\omega L}{c_1}} + \mathcal{R}_0}{1 + \mathcal{R}_0\mathcal{R}_1 e^{2i\frac{\omega L}{c_1}}},$$

$$\hat{\mathcal{T}}(\omega) = \frac{1}{\overline{\hat{U}(\omega)}} = \frac{\mathcal{T}_0\mathcal{T}_1 e^{i\frac{\omega L}{c_1}}}{1 + \mathcal{R}_0\mathcal{R}_1 e^{2i\frac{\omega L}{c_1}}},$$

and $\mathcal{R}_j$ and $\mathcal{T}_j$ are the reflection and transmission coefficients of interface j.

Numerical illustration of reflection and transmission on a homogeneous slab

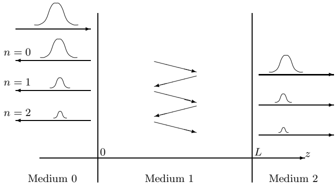
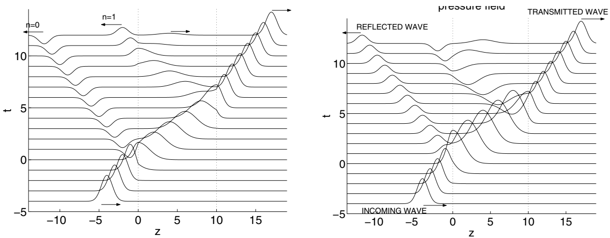


FIGURE – Sketch of scattering sequence for a wave incoming from the left on a homogeneous slab.



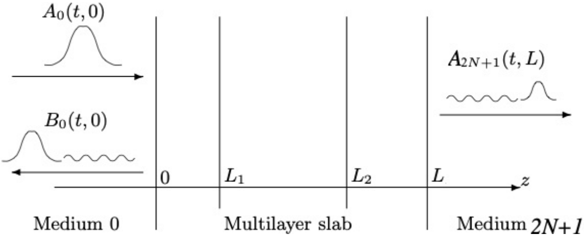
(a) Pressure $p(t, z)$ (b) Velocity $u(t, z)$

FIGURE – Waves scattered by a homogeneous slab from a wave incoming from the left.

Outline

1. Acoustic wave propagation in homogeneous and simple heterogeneous media
   - Acoustic wave propagation in homogeneous media
   - Scattering by a single interface
   - Scattering by a homogeneous slab
2. Effective properties of a finely-layered slab
   - Periodic case with two materials
   - Stochastic case with two materials
   - General stochastic case
3. Transmission of energy in 1D in the weakly heterogeneous regime
   - Transmission of monochromatic waves
   - A result in asymptotic analysis of random ODEs
   - Exponential decay of energy and localization
4. Beyond the 1D weak scattering approximation
   - Transmission of energy in the strongly heterogeneous regime
   - 3D Wave propagation in a randomly-layered medium

Effective properties of a finely-layered slab



We consider a multi-layered with $2N$ layers in-between two homogeneous half-spaces

$$(\rho, K)(z) = \begin{cases} (\rho_e, K_e) & \text{if } z < L_0 \\ (\rho_a, K_a) & \text{if } z \in [L_{2j}, L_{2j+1}], \quad j = 0, 1, ..., N-1 \\ (\rho_b, K_b) & \text{if } z \in [L_{2j+1}, L_{2(j+1)}], \quad j = 0, 1, ..., N-1 \\ (\rho_{e'}, K_{e'}) & \text{if } z > L_{2N} \end{cases},$$

We introduce the thicknesses and time lags, $j \geq 1$:

$$\Delta_j = L_j - L_{j-1}$$

Interface propagator for a pair of interfaces

The scattering problem is set as

$$\begin{bmatrix} \hat{a}_{2N+1}(\omega) \\ 0 \end{bmatrix} = \hat{\mathbf{K}}_{2N}(\omega) \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(\omega) \end{bmatrix}, \quad \hat{\mathbf{K}}_{2N}(\omega) = \hat{\mathbf{J}}_{2N}(\omega)\hat{\mathbf{J}}_{2N-1}(\omega) \left(\prod_{j=1}^{N-1} \hat{\mathbf{J}}_{2j}(\omega)\hat{\mathbf{J}}_{2j-1}(\omega) \right) \mathbf{J}_0.$$

The interface propagator for a pair of two successive interfaces is

$$\begin{aligned} \hat{\mathbf{J}}_{2j}(\omega)\hat{\mathbf{J}}_{2j-1}(\omega) &= \begin{bmatrix} r^{(+)} e^{i\omega \frac{\Delta_{2j}}{c_b}} & -r^{(-)} e^{-i\omega \frac{\Delta_{2j}}{c_b}} \\ -r^{(-)} e^{i\omega \frac{\Delta_{2j}}{c_b}} & r^{(+)} e^{-i\omega \frac{\Delta_{2j}}{c_b}} \end{bmatrix} \begin{bmatrix} r^{(+)} e^{i\omega \frac{\Delta_{2j-1}}{c_a}} & r^{(-)} e^{-i\omega \frac{\Delta_{2j-1}}{c_a}} \\ r^{(-)} e^{i\omega \frac{\Delta_{2j-1}}{c_a}} & r^{(+)} e^{-i\omega \frac{\Delta_{2j-1}}{c_a}} \end{bmatrix} \\ &= \begin{bmatrix} e^{i\omega \left(\frac{\Delta_{2j-1}}{c_a} + \frac{\Delta_{2j}}{c_b} \right)} & 0 \\ 0 & e^{-i\omega \left(\frac{\Delta_{2j-1}}{c_a} + \frac{\Delta_{2j}}{c_b} \right)} \end{bmatrix} + r^{(-)} \begin{bmatrix} r^{(-)} \delta_j^{(+)} & -r^{(+)} \delta_j^{(-)} \\ -r^{(+)} \delta_j^{(+)} & r^{(-)} \delta_j^{(-)} \end{bmatrix} \end{aligned}$$

where $r^{(\pm)} = \left(\sqrt{\zeta_b/\zeta_a} \pm \sqrt{\zeta_a/\zeta_b} \right)/2$, with $r^{(+)2} - r^{(-)2} = 1$ and

$$\delta_j^{(\pm)} = \pm 2 e^{\pm i\omega \frac{\Delta_{2j-1}}{c_a}} \sin\left(\omega \frac{\Delta_{2j}}{c_b} \right)$$

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Effective properties of a finely-layered slab : periodic case

We now assume a periodic case, with all layers of the same thickness, $1 \leq j \leq N$

$$\Delta_{2j-1} = \Delta_{2j} = \Delta, \quad \left(\frac{1}{c_a} + \frac{1}{c_b}\right) = \frac{2}{\overline{c}}, \quad \delta_j^{(\pm)} = \pm 2\omega \frac{\Delta}{c_b} + \mathcal{O}(\Delta^2)$$

and

$$\hat{\mathbf{K}}_{2N}(\omega) = \hat{\mathbf{J}}_{2N}(\omega, \Delta)\hat{\mathbf{J}}_{2N-1}(\omega, \Delta) \left(\hat{\mathbf{J}}^{(2)}(\omega, \Delta)\right)^{N-1} \mathbf{J}_0.$$

where

$$\hat{\mathbf{J}}^{(2)}(\omega, \Delta) = \mathbf{I}_2 + 2i\omega\Delta \begin{bmatrix} \frac{1}{\overline{c}} + \frac{r^{(-)2}}{c_b} & \frac{r^{(+)}r^{(-)}}{c_b} \\ -\frac{r^{(+)}r^{(-)}}{c_b} & -\frac{1}{\overline{c}} - \frac{r^{(-)2}}{c_b} \end{bmatrix} + \mathcal{O}\left(\Delta^2\right)$$

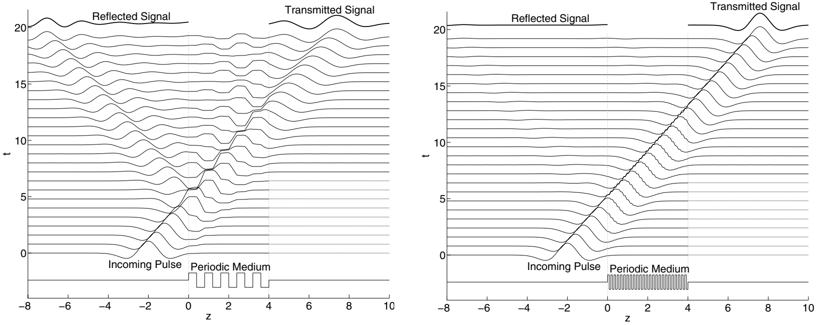
and

$$\lim_{\Delta \to 0} \left(\hat{\mathbf{J}}^{(2)}(\omega, \Delta)\right)^{N-1} = \begin{bmatrix} r_{b*}^{(+)} & r_{b*}^{(-)} \\ r_{b*}^{(-)} & r_{b*}^{(+)} \end{bmatrix} \begin{bmatrix} e^{i\omega \frac{L}{\overline{c}}} & 0 \\ 0 & e^{-i\omega \frac{L}{\overline{c}}} \end{bmatrix} \begin{bmatrix} r_{*a}^{(+)} & r_{*a}^{(-)} \\ r_{*a}^{(-)} & r_{*a}^{(+)} \end{bmatrix}$$

Eventually (using also that $\mathbf{J}_{2N}(\omega, \Delta)\mathbf{J}_{2N-1}(\omega, \Delta) \to_{\Delta \to 0} \mathbf{J}_{e'b}$)

$$\lim_{\Delta \to 0} \hat{\mathbf{K}}_{2N+1}(\omega) = \mathbf{J}_{e'*} \left(\hat{\mathbf{J}}^{(2)}(\omega, \Delta)\right)^{N} \mathbf{J}_{*e}.$$

Numerical illustration of a wave impacting a periodic slab



(a) Large period $\lambda \approx \ell_c$ (b) Short period $\lambda \gg \ell_c$

FIGURE – Waves generated around periodic slabs with different periods by a wave incoming from the left. The half-spaces have the same properties as the homogenized medium.

- In the limit $\Delta \to 0$, we retrieve a homogeneous equation
- When the wave enters (resp. exits) the slab, it interacts with the homogenized medium, NOT the first (resp. last) layer.

Outline

Effective properties of a randomly-layered slab with two materials

We assume the layer thickness is now a random variable

$$\Delta_j = \delta U_j$$

where $\delta > 0$ and the U_j are independent and identically distributed uniform random variables over $[1/2, 3/2]$. The thickness of the slab is a random variable L with

$$\mathbb{E}[L] = 2N\delta = \overline{L}, \quad \mathbb{E}\left[(L-\overline{L})^2\right] = \frac{2N\delta^2}{12} \to_{\delta\to 0} 0$$

The scattering problem yields (again grouping the layers by pairs)

$$\hat{\mathbf{K}}_{2N}(\omega) = \hat{\mathbf{J}}_{2N}(\omega,\Delta)\hat{\mathbf{J}}_{2N-1}(\omega,\Delta)\left(\hat{\mathbf{J}}^{(2)}(\omega,\Delta)\right)^{N-1}\mathbf{J}_0.$$

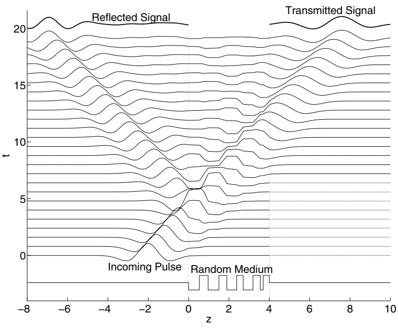
where Taylor expansion still can be used $\hat{\mathbf{J}}^{(2)}(\omega,\delta) = \mathbf{I}_2 + i\omega\delta\hat{\mathbf{J}}^{(2)}_{j,1} + \mathcal{O}\left(\delta^2\right)$, where

$$\hat{\mathbf{J}}^{(2)}_{j,1} = \begin{bmatrix} r^{(+)2}\left(\frac{U_{2j-1}}{c_a}+\frac{U_{2j}}{c_b}\right) - r^{(-)2}\left(\frac{U_{2j-1}}{c_a}-\frac{U_{2j}}{c_b}\right) & 2r^{(+)}r^{(-)}\frac{U_{2j}}{c_b} \\ -2r^{(+)}r^{(-)}\frac{U_{2j}}{c_b} & -r^{(+)2}\left(\frac{U_{2j-1}}{c_a}+\frac{U_{2j}}{c_b}\right) + r^{(-)2}\left(\frac{U_{2j-1}}{c_a}-\frac{U_{2j}}{c_b}\right) \end{bmatrix}$$

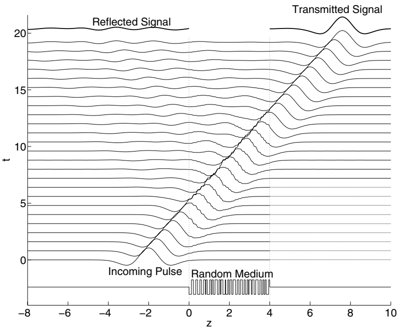
is a random matrix, independent of δ and the law of large numbers yields

$$\delta\sum_{j=1}^{N}\hat{\mathbf{J}}^{(2)}_{j,1} \to_{N\to\infty} \delta N\mathbb{E}\left[\hat{\mathbf{J}}^{(2)}_{j,1}\right] = \frac{\overline{L}}{2}\hat{\mathbf{J}}^{(2)}_{1}$$

Numerical illustration of a wave impacting a slab with random thicknesses



(a) Large period $\lambda \approx \ell_c$



(b) Short period $\lambda \gg \ell_c$

FIGURE – Waves generated around a random slabs with different periods by a wave incoming from the left. The half-spaces have the same properties as the homogenized medium.

- The result is qualitatively the same as in a periodic medium.

Outline

Effective properties of a finely-layered slab : general stochastic case

The pressure $p(t, z)$ and velocity $u(t, z)$ in a heterogeneous medium with density $\rho(z)$ and bulk modulus $K(z)$ verify

$$\rho(z)\frac{\partial u(t,z)}{\partial t} + \frac{\partial p(t,z)}{\partial z} = 0,$$
$$\frac{1}{K(z)}\frac{\partial p(t,z)}{\partial t} + \frac{\partial u(t,z)}{\partial z} = 0.$$

We consider a continuously-fluctuating heterogeneous slab of thickness L in-between two homogeneous half-spaces

$$\rho(z) = \begin{cases} \rho_0 & \text{if } z < 0 \\ \rho(z/\ell) & \text{if } z \in [0, L] \\ \rho_1 & \text{if } z > L \end{cases}, \quad K(z) = \begin{cases} K_0 & \text{if } z < 0 \\ K(z/\ell) & \text{if } z \in [0, L] \\ K_1 & \text{if } z > L \end{cases}.$$

where ℓ is a characteristic scale of fluctuations in the heterogeneous medium (typically a correlation length in a random medium).

Boundary conditions

In the half-spaces, the solution is decomposed as right- and left-going waves

$$A_0(t,z) = \zeta_0^{-1/2} p(t,z) + \zeta_0^{1/2} u(t,z), \quad B_0(t,z) = -\zeta_0^{-1/2} p(t,z) + \zeta_0^{1/2} u(t,z), \quad z < 0$$

where $c_0(z) = \sqrt{K_0(z)/\rho_0(z)}$ and $\zeta_0(z) = \sqrt{K_0(z)\rho_0(z)}$, and

$$A_1(t,z) = \zeta_1^{-1/2} p(t,z) + \zeta_1^{1/2} u(t,z), \quad B_1(t,z) = -\zeta_1^{-1/2} p(t,z) + \zeta_1^{1/2} u(t,z), \quad z > L$$

where $c_1(z) = \sqrt{K_1(z)/\rho_1(z)}$ and $\zeta_1(z) = \sqrt{K_1(z)\rho_1(z)}$.
We are still interested in the scattering of an incoming wave from the left

$$A_0(t,z) = f\left(t - \frac{z}{c_0}\right), \quad B_1(t,z) = 0$$

Projection along constant characteristics in the heterogeneous slab

Within the heterogeneous slab, we decompose the field along right- and left-going waves of an homogenized medium (whose precise parameters $\overline{c}$ and $\overline{\zeta}$ will be discussed later)

$$A = \overline{\zeta}^{-1/2} p + \overline{\zeta}^{1/2} u, \quad B = -\overline{\zeta}^{-1/2} p + \overline{\zeta}^{1/2} u$$

Using the equilibrium equation for (u, p), this decomposition yields

$$\frac{\partial A}{\partial z} = \frac{1}{\overline{\zeta}^{1/2}} \frac{\partial p}{\partial z} + \overline{\zeta}^{1/2} \frac{\partial u}{\partial z} = -\frac{1}{\overline{c}} \left(\Delta_\ell^{(+)} \frac{\partial A}{\partial z} + \Delta_\ell^{(-)} \frac{\partial B}{\partial z} \right)$$

where

$$\Delta_\ell^{(\pm)}(z) = \Delta^{(\pm)} \left(\frac{z}{\ell} \right) = \frac{1}{2} \left(\frac{\rho(z/\ell)}{\overline{\rho}} \pm \frac{\overline{K}}{K(z/\ell)} \right)$$

A similar decomposition starting from $\partial B / \partial z$ yields the system

$$\frac{\partial}{\partial z} \begin{bmatrix} A \\ B \end{bmatrix} = -\frac{1}{\overline{c}} \begin{bmatrix} \Delta_\ell^{(+)} & \Delta_\ell^{(-)} \\ -\Delta_\ell^{(-)} & -\Delta_\ell^{(+)} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A \\ B \end{bmatrix}.$$

Contrarily to the homogeneous case, the right- and left-going modes are now coupled by the heterogeneities.

Centering of the modes

We consider moving frames following the right- and left-going modes.

$$a(s,z) = A\left(s + \frac{z}{\overline{c}}, z\right), \quad b(s,z) = B\left(s - \frac{z}{\overline{c}}, z\right)$$

In a 1D homogeneous medium, the solutions do not really depend on s because in the moving frames, the shapes are constant.
In Fourier space (and considering ω as a parameter)

$$\begin{aligned}
\frac{d\hat{a}}{dz} &= \int_{\mathbb{R}} e^{i\omega s} \frac{\partial}{\partial z} A\left(s + \frac{z}{\overline{c}}, z\right) ds = \int_{\mathbb{R}} e^{i\omega s} \left(\frac{1}{\overline{c}} \frac{\partial A}{\partial s}\left(s + \frac{z}{\overline{c}}, z\right) + \frac{\partial A}{\partial z}\left(s + \frac{z}{\overline{c}}, z\right)\right) ds \\
&= -\frac{i\omega}{\overline{c}} \int_{\mathbb{R}} e^{i\omega s} \left(a(s,z) - \Delta_\ell^{(+)}(z) a(s,z) - \Delta_\ell^{(-)}(z) b\left(s + \frac{2z}{\overline{c}}, z\right)\right) ds \\
&= -\frac{i\omega}{\overline{c}} \left(\left(1 - \Delta_\ell^{(+)}(z)\right) \hat{a}(\omega,z) - \Delta_\ell^{(-)}(z) e^{-2i\omega z/\overline{c}} \hat{b}(\omega,z)\right)
\end{aligned}$$

Eventually, with similar transformation for the left-going mode

$$\frac{d}{dz}\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix} = \frac{i\omega}{\overline{c}} \begin{bmatrix} -\left(1 - \Delta_\ell^{(+)}(z)\right) & \Delta_\ell^{(-)}(z) e^{-2i\omega z/\overline{c}} \\ -\Delta_\ell^{(-)}(z) e^{2i\omega z/\overline{c}} & \left(1 - \Delta_\ell^{(+)}(z)\right) \end{bmatrix} \begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}.$$

Transforming a BVP into an IVP

To transform the formulation into an IVP, we introduce the propagator $\mathbf{P}_\omega$, a 2×2 matrix solution of the IVP

$$\frac{d}{dz}\mathbf{P}_\omega(z) = \mathbf{H}_\omega\left(z, \frac{z}{\ell}\right)\mathbf{P}_\omega(z), \quad \mathbf{P}_\omega(0) = \mathbf{I}_2$$

where

$$\mathbf{H}_\omega(z, z') = \frac{i\omega}{\overline{c}}\begin{bmatrix} -\left(1 - \Delta^{(+)}(z')\right) & \Delta^{(-)}(z')e^{-2i\omega z/\overline{c}} \\ -\Delta^{(-)}(z')e^{2i\omega z/\overline{c}} & \left(1 - \Delta^{(+)}(z')\right) \end{bmatrix}$$

depends on both the slow (z) and fast (z') variables
The propagator propagates the solution to all positions z given the solution in $z = 0$

$$\begin{bmatrix} \hat{a}(\omega, z) \\ \hat{b}(\omega, z) \end{bmatrix} = \mathbf{P}_\omega(z)\begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}.$$

Averaging theorem

Now that the system is under an appropriate form, we can use (a first basic form of) an averaging theorem for systems of random differential equations

Theorem 1 (A "qualitative" averaging theorem)

Let Y be an ergodic random process and $F(z, Y, X)$ a smooth function that increases at most linearly in X. The solution X of the random differential equation

$$\frac{dX}{dz} = F\left(z, Y\left(\frac{z}{\ell}\right), X(z)\right), \quad X(0) = x_0$$

converges for small ℓ to the solution $\overline{X}$ of the deterministic differential equation

$$\frac{d\overline{X}}{dz} = \overline{F}\left(z, \overline{X}(z)\right), \quad \overline{X}(0) = x_0$$

where

$$\overline{F}(z, x) = \lim_{L \to \infty} \frac{1}{L} \int_0^L F(z, Y(y), x) dy = \mathbb{E}[F(z, Y, x)]$$

Note that the small scale ℓ only enters through the driving random process Y and the expectation for F is taken with x and z fixed.

Main steps of the "proof"

1. We have

$$\begin{aligned} X(z) - \overline{X}(z) &= \int_0^z F\left(y, Y\left(\frac{y}{\ell}\right), X(y)\right) dy - \int_0^z \overline{F}\left(y, \overline{X}(y)\right) dy \\ &= \int_0^z \left(F\left(y, Y\left(\frac{y}{\ell}\right), X(y)\right) - F\left(y, Y\left(\frac{y}{\ell}\right), \overline{X}(y)\right)\right) dy + g(z) \end{aligned}$$

   with

$$g(x) = \int_0^z \left(F\left(y, Y\left(\frac{y}{\ell}\right), \overline{X}(y)\right) - \overline{F}\left(y, \overline{X}(y)\right)\right) dy$$

   With some assumptions on the regularity of F (with respect to X), we have

$$\left|X(z) - \overline{X}(z)\right| \leq C \int_0^z \left|X(y) - \overline{X}(y)\right| dy + |g(x)|$$

2. The law of large numbers implies that $\lim_{\ell \to 0} |g(x)| = 0$.
3. Finally, Gronwall's lemma states that, for $t \geq 0$ and $A, B \geq 0$,

$$Z(t) \leq A + B \int_0^t Z(s) ds \implies Z(t) \leq A e^{Bt},$$

   which allows to conclude.

Application of the averaging theorem to our problem

We have $Y(z) = (\rho(z), K(z))$ and

$$\Delta_\ell^{(\pm)}(z) = \Delta^{(\pm)}\left(\frac{z}{\ell}\right) = \frac{1}{2}\left(\frac{\rho(z/\ell)}{\overline{\rho}} \pm \frac{\overline{K}}{K(z/\ell)}\right)$$

so that, choosing $\overline{\rho} = \mathbb{E}[\rho]$ and $\overline{K} = \mathbb{E}[1/K]^{-1}$, we have

$$\mathbb{E}[\Delta_\ell^{(+)}(z)] = 1, \quad \mathbb{E}[\Delta_\ell^{(-)}(z)] = 0$$

The right-hand side of the propagator equation is $\mathbf{F}_\omega(z, Y(z'), X(z)) = \mathbf{H}_\omega(z, z')\mathbf{P}_\omega(z)$, with

$$\mathbf{H}_\omega(z, z') = \frac{i\omega}{\overline{c}}\begin{bmatrix} -\left(1 - \Delta^{(+)}(z')\right) & -\Delta^{(-)}(z')e^{-2i\omega z/\overline{c}} \\ \Delta^{(-)}(z')e^{2i\omega z/\overline{c}} & \left(1 - \Delta^{(+)}(z')\right) \end{bmatrix}$$

so that the averaging theorem predicts that the homogenized solution $\overline{\mathbf{P}}$ verifies

$$\frac{d\overline{\mathbf{P}}_\omega(z)}{dz} = \mathbf{0}, \quad \overline{\mathbf{P}}(0) = \mathbf{I}_2$$

so that $\overline{\mathbf{P}}_\omega(z) = \mathbf{I}_2$ at all positions, which is the result of a wave equation in a homogeneous medium (remember the rescaling of the right- and left-going modes ! !).

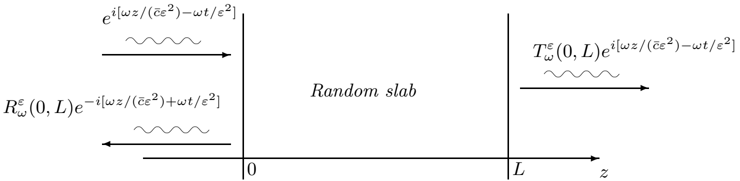
Discussion

- The bound that we used in the "proof" may diverge at large times ?
- The homogenized solution of the previous section was very particular in the sense that it was deterministic. Often, only functionals (quantities of interest) will be deterministic but not the solution itself.
- In this next part, we consider a case where the solution is stochastic, but the norm of the transmission coefficient $|\mathcal{T}(\omega)|^2$ is deterministic

Outline

Outline

Transmission of energy through a slab of random medium



We consider a continuously-fluctuating heterogeneous slab of thickness L in-between two homogeneous half-spaces

$$\rho(z) = \bar{\rho}, \quad \frac{1}{K(z)} = \begin{cases} \frac{1}{\bar{K}} & \text{if } z < 0 \\ \frac{1}{\bar{K}}\left(1 + \epsilon\nu\left(\frac{z}{\epsilon^2}\right)\right) & \text{if } z \in [0, L] \\ \frac{1}{\bar{K}} & \text{if } z > L \end{cases} .$$

where $\nu(z) = g(Y(z))$ is a zero-mean, stationary random process, and Y is a homogeneous in z Markov process (other technical hypotheses required). We consider the weakly-heterogeneous scaling regime, and the frequency of the incoming wave is ω/ϵ^2.

The boundary value problems for right- and left-going modes

The setting is exactly the same as in the previous part, so we retrieve the equation for the centered right- and left-going modes (at frequency ω/ϵ^2) :

$$\frac{d}{dz}\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix} = \frac{i\omega}{\overline{c}\epsilon^2}\begin{bmatrix}-\left(1-\Delta^{(+)}(z)\right) & -\Delta^{(-)}(z)e^{-2i\omega z/\overline{c}\epsilon^2} \\ \Delta^{(-)}(z)e^{2i\omega z/\overline{c}\epsilon^2} & \left(1-\Delta^{(+)}(z)\right)\end{bmatrix}\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix}.$$

where

$$\Delta^{(\pm)}(z) = \frac{1}{2}\left(\frac{\rho(z)}{\overline{\rho}} \pm \frac{\overline{K}}{K(z)}\right) = \frac{1}{2}\left(1 \pm 1 \pm \epsilon\nu\left(\frac{z}{\epsilon^2}\right)\right)$$

Eventually, the modes verify

$$\frac{d}{dz}\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix} = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon^2}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix}.$$

with

$$\mathbf{H}_\omega(z,\nu) = \frac{i\omega}{2\overline{c}}\nu\begin{bmatrix}1 & -e^{-2i\omega z/\overline{c}} \\ e^{2i\omega z/\overline{c}} & -1\end{bmatrix}$$

and boundary conditions at the interfaces with the half-spaces.

Properties and parameterization of the propagator

As earlier, we introduce the propagator $\mathbf{P}_\omega(z)$, the 2×2 matrix solution of the IVP

$$\frac{d}{dz}\mathbf{P}_\omega(z) = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon^2}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\mathbf{P}_\omega(z), \quad \mathbf{P}_\omega(0) = \mathbf{I}_2$$

Note that the function $\mathbf{H}_\omega$ depends on ϵ through both arguments so the previous averaging theorem cannot be used.
We observe that if $[\alpha_\omega, \beta_\omega]^T$ is a solution of the propagator equation with initial value $[1, 0]^T$, then the properties of $\mathbf{H}_\omega$ imply that

$$\frac{d}{dz}\begin{bmatrix}\overline{\beta}_\omega(z) \\ \overline{\alpha}_\omega(z)\end{bmatrix} = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon^2}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\begin{bmatrix}\overline{\beta}_\omega(z) \\ \overline{\alpha}_\omega(z)\end{bmatrix}, \quad \text{with} \quad \begin{bmatrix}\overline{\beta}_\omega(0) \\ \overline{\alpha}_\omega(0)\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix}$$

so we can parameterize the propagator as

$$\mathbf{P}_\omega = \begin{bmatrix}\alpha_\omega & \overline{\beta}_\omega \\ \beta_\omega & \overline{\alpha}_\omega\end{bmatrix}$$

Using Jacobi's formula for the derivative of a determinant

$$\frac{d}{dz}\det\mathbf{P}_\omega = \mathrm{Tr}\left((\det\mathbf{P}_\omega)\mathbf{P}_\omega^{-1}\frac{d}{dz}\mathbf{P}_\omega\right) = \det\mathbf{P}_\omega\mathrm{Tr}\left(\mathbf{P}_\omega^{-1}\mathbf{H}_\omega\mathbf{P}_\omega\right) = 0$$

The initial condition yields $\det\mathbf{P}_\omega = 1$ and $|\alpha_\omega(z)|^2 - |\beta_\omega(z)|^2 = 1$, for all $z \geq 0$.

Quantities of interest

We are still interested in the reflection and transmission coefficients, now in the frequency domain, for a unit input load $\hat{f}(\omega) = 1$,

$$\hat{\mathcal{R}}^{\epsilon}_{\omega} = \hat{b}^{\epsilon}_{\omega}(0), \quad \hat{\mathcal{T}}^{\epsilon}_{\omega} = \hat{a}^{\epsilon}_{\omega}(L)$$

Since we considered a matched medium,

$$\begin{bmatrix} \hat{\mathcal{T}}(\omega) \\ 0 \end{bmatrix} = \mathbf{P}_{\omega}(L) \begin{bmatrix} 1 \\ \hat{\mathcal{R}}(\omega) \end{bmatrix},$$

we obtain

$$\hat{\mathcal{R}}(\omega) = -\frac{\beta_{\omega}}{\overline{\alpha}_{\omega}}, \quad \hat{\mathcal{T}}(\omega) = \frac{1}{\overline{\alpha}_{\omega}}$$

Outline

A result in asymptotic analysis of random ODEs

Theorem 2 (Averaging in the weakly heterogeneous scaling regime)

Let $X^\epsilon(z)$ for $z \geq 0$ be the process in $\mathbb{R}^d$ defined by the random differential equation

$$\frac{dX^\epsilon}{dz}(z) = \frac{1}{\epsilon} F\left(X^\epsilon(z), Y\left(\frac{z}{\epsilon^2}\right), \frac{z}{\epsilon^2}\right),$$

starting from $X^\epsilon(0) = x_0 \in \mathbb{R}^d$. Assume that $Y(z)$ is a z-homogeneous Markov ergodic process (other technical hypotheses required), and the $\mathbb{R}^d$-valued function F, periodic with respect to τ of period Z_0, satisfies the centering condition $\int_0^{Z_0} \mathbb{E}[F(x, Y(0), \tau)]d\tau = 0$. Assume also that $F(x, y)$ is at most linearly growing and smooth in x. Then the random processes $X^\epsilon(z)$ converge in distribution to the Markov diffusion process $X(z)$ with generator :

$$\mathcal{L}\phi = \frac{1}{Z_0} \int_0^{Z_0} \int_0^\infty \mathbb{E}\left[F(x, Y(0), \tau) \cdot \nabla(F(x, Y(z), \tau + z) \cdot \nabla\phi)\right] dz d\tau$$

Proof : see [2]

2. J.-P. Fouque et al. *Wave propagation and time reversal in randomly layered media*. T. 56. Stochastic Modelling and Applied Probability. Springer, 2007.

Computational aspects for a particular form of the functional F

We assume additionally that the variable X^ϵ is a $d \times d$-matrix and the functional has the specific form

$$\mathbf{F}(\mathbf{P}, y, \tau) = \sum_{p=1}^{n} g^{(p)}(y, \tau)\mathbf{h}^{(p)}\mathbf{P}$$

where $\mathbf{h}^{(p)}$ are constant matrices, $g^{(p)}(y, \tau)$ are real-valued scalar functions of $y \in S$ and $\tau \in \mathbb{R}$ verifying $\mathbb{E}[g^{(p)}(Y(0), \tau)] = 0$. Then the limit generator is

$$\mathcal{L}\phi(\mathbf{P}) = \frac{1}{2}\sum_{p,q=1}^{n} C_{pq}\left(\mathbf{h}^{(p)}\mathbf{P}\right)\cdot\nabla\left(\mathbf{h}^{(q)}\mathbf{P}\cdot\nabla\phi(\mathbf{P})\right)$$

where

$$C_{pq} = 2\frac{1}{Z_0}\int_0^{Z_0}\int_0^{\infty}\mathbb{E}[g^{(p)}(Y(0), \tau)g^{(q)}(Y(z), \tau + z)]dzd\tau$$

We also introduce the square root of the symmetric part of the correlation $\tilde{\boldsymbol{\sigma}}^2 = \mathbf{C}^{(S)}$, and

$$\tilde{\mathbf{h}}_\ell = \sum_{p=1}^{n}\tilde{\sigma}_{\ell p}\mathbf{h}_p.$$

Then the limit diffusion process $\mathbf{P}(z)$ is identified with the solution of the Stratonovich stochastic differential equation

$$d\mathbf{P}(z) = \sum_{\ell=1}^{n}\tilde{\mathbf{h}}_\ell\mathbf{P}(z)\circ dW_\ell(z) + \frac{1}{2}\sum_{p,q=1}^{n} C_{pq}^{(A)}\mathbf{h}_q\mathbf{h}_p\mathbf{P}(z)dz, \quad \mathbf{P}(0) = \mathbf{I}_2$$

where the $W_\ell(z)$ are independent standard Brownian motions.

Outline

Back to the transmission of energy through the random slab

We are interested in the limit process (for $\epsilon \to 0$) of the propagator equation

$$\frac{d}{dz}\mathbf{P}_\omega(z) = \frac{1}{\epsilon}\mathbf{F}_\omega\left(\mathbf{P}_\omega, \nu\left(\frac{z}{\epsilon^2}\right)\frac{z}{\epsilon^2}\right) = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon^2}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\mathbf{P}_\omega(z), \quad \mathbf{P}_\omega(0) = \mathbf{I}_2$$

where

$$\begin{aligned}\mathbf{F}_\omega(\mathbf{P}_\omega, \nu, z) &= \frac{i\omega}{2\overline{c}}\nu\begin{bmatrix}1 & -e^{-2i\omega z/\overline{c}}\\ e^{2i\omega z/\overline{c}} & -1\end{bmatrix}\mathbf{P}_\omega \\ &= \frac{i\omega}{2\overline{c}}\nu\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\mathbf{P}_\omega - \frac{\omega}{2\overline{c}}\nu\sin\left(\frac{2\omega z}{\overline{c}}\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{P}_\omega - \frac{i\omega}{2\overline{c}}\nu\cos\left(\frac{2\omega z}{\overline{c}}\right)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{P}_\omega\end{aligned}$$

We identify $n = 3$, $Z_0 = \pi\overline{c}/\omega$

$$g^{(1)}(\nu, \tau) = \nu, \quad g^{(2)}(\nu, \tau) = \nu\sin\left(\frac{2\omega\tau}{\overline{c}}\right), \quad g^{(3)}(\nu, \tau) = \nu\cos\left(\frac{2\omega z}{\overline{c}}\right)$$

$$\mathbf{h}^{(1)} = \frac{i\omega}{2\overline{c}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \mathbf{h}^{(2)} = -\frac{\omega}{2\overline{c}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \mathbf{h}^{(3)} = \frac{\omega}{2\overline{c}}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

The covariance matrix

We have

$$C_{pq} = 2\frac{1}{Z_0}\int_0^{Z_0}\int_0^{\infty}\mathbb{E}[g^{(p)}(Y(0),\tau)g^{(q)}(Y(z),\tau+z)]dzd\tau$$

and obtain

$$\mathbf{C} = \begin{bmatrix} \gamma(0) & 0 & 0 \\ 0 & \frac{1}{2}\gamma(\omega) & -\frac{1}{2}\gamma^{(s)}(\omega) \\ 0 & \frac{1}{2}\gamma^{(s)}(\omega) & \frac{1}{2}\gamma(\omega) \end{bmatrix},$$

as a function of the covariance of the bulk modulus

$$\gamma(\omega) = 2\int_0^{\infty}\mathbb{E}[\nu(0)\nu(z)]\cos\left(\frac{2\omega z}{\overline{c}}\right)dz$$

$$\gamma^{(s)}(\omega) = 2\int_0^{\infty}\mathbb{E}[\nu(0)\nu(z)]\sin\left(\frac{2\omega z}{\overline{c}}\right)dz$$

In these functions, the scaling of wavelength $\overline{\lambda} = 2\pi\overline{c}/\omega$ and correlation length is important to drive the interaction of the wave with the fluctuations.
Additionally

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \sqrt{\gamma(0)} & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}\gamma(\omega)} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}\gamma(\omega)} \end{bmatrix}, \quad \mathbf{C}^{(A)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\gamma^{(s)}(\omega) \\ 0 & \frac{1}{2}\gamma^{(s)}(\omega) & 0 \end{bmatrix}.$$

More ingredients

Then the limit diffusion process $\mathbf{P}(z)$ is identified with the solution of the Stratonovich stochastic differential equation

$$d\mathbf{P}(z) = \sum_{\ell=1}^{n} \tilde{\sigma}_{\ell\ell}\mathbf{h}^{(\ell)}\mathbf{P}(z) \circ dW_\ell(z) + \frac{1}{2}\sum_{p,q=1}^{n} C_{pq}^{(A)}\mathbf{h}_q\mathbf{h}_p\mathbf{P}(z)dz, \quad X_i(0) = x_{0,i}$$

where the $W_\ell(z)$ are independent standard Brownian motions.
More in details :

$$d\mathbf{P}(z) = \frac{i\omega\sqrt{\gamma(0)}}{2\overline{c}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\mathbf{P}(z)\circ dW_1(z) - \frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\overline{c}}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\mathbf{P}(z)\circ dW_2(z)$$
$$- \frac{i\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\overline{c}}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\mathbf{P}(z)\circ dW_3(z) - \frac{i\omega^2\gamma^{(s)}(\omega)}{8\overline{c}^2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\mathbf{P}(z)\circ dz$$

or, in Itô form

$$d\mathbf{P}(z) = \frac{i\omega\sqrt{\gamma(0)}}{2\overline{c}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\mathbf{P}(z)dW_1(z) - \frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\overline{c}}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\mathbf{P}(z)dW_2(z)$$
$$- \frac{i\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\overline{c}}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\mathbf{P}(z)dW_3(z) - \frac{i\omega^2\gamma^{(s)}(\omega)}{8\overline{c}^2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\mathbf{P}(z)dz - \frac{\omega^2(\gamma(0)-\gamma(\omega))}{8\overline{c}^2}\mathbf{P}(z)dz$$

Limit process for transmission through a random slab

We are interested in the transmission coefficient

$$|\hat{\mathcal{T}}(\omega)|^2 = \frac{1}{|\overline{\alpha}_\omega|^2}$$

After (lengthy) computations, one obtains that, with probability one

$$\lim_{L\to\infty} \frac{1}{L} \ln\left(|\hat{\mathcal{T}}(\omega)|^2\right) = -\frac{1}{L_{\mathrm{loc}}}$$

where the so-called localization length is defined as

$$L_{\mathrm{loc}}(\omega) = \frac{4\overline{c}^2}{\gamma(\omega)\omega^2}.$$

More precisely, one can show that

$$\lim_{L\to\infty} |\hat{\mathcal{T}}(\omega)|^2 = \exp\left(-\frac{L}{L_{\mathrm{loc}}(\omega)} - \sqrt{\frac{2}{L_{\mathrm{loc}}(\omega)}}W^*(L)\right)$$

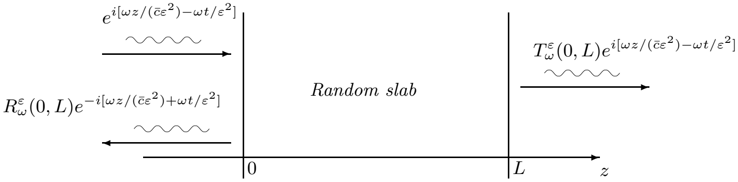
where $W^*(z)$ is a standard Brownian motion.
It is a rather extraordinary effect that (with 1D heterogeneity only) the decay of the transmission is exponential, whatever the strength of fluctuations. This is Anderson localization [3].

3. P. W. Anderson. "Absence of diffusion in certain random lattices". In : *Phys. Rev.* 109.5 (1958), p. 1492-1505. DOI : 10.1103/PhysRev.109.1492

Outline

Outline

Transmission of energy in the strongly heterogeneous regime



We consider the same setting as earlier, in the strongly scattering regime, where the frequency of the incoming wave is ω/ϵ.

$$\rho(z) = \overline{\rho}, \quad \frac{1}{K(z)} = \begin{cases} \frac{1}{\overline{K}} & \text{if } z < 0 \\ \frac{1}{\overline{K}}\left(1 + \epsilon^0 \nu\left(\frac{z}{\epsilon^2}\right)\right) & \text{if } z \in [0, L] \\ \frac{1}{\overline{K}} & \text{if } z > L \end{cases} .$$

Propagator equation in the strong scattering regime

The setting is exactly the same as in the previous part, so we retrieve the equations for the centered right- and left-going modes (at frequency ω/ϵ) :

$$\frac{d}{dz}\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix} = \frac{i\omega}{\overline{c}\epsilon}\begin{bmatrix} -\left(1-\Delta^{(+)}(z)\right) & -\Delta^{(-)}(z)e^{-2i\omega z/\overline{c}\epsilon} \\ \Delta^{(-)}(z)e^{2i\omega z/\overline{c}\epsilon} & \left(1-\Delta^{(+)}(z)\right)\end{bmatrix}\begin{bmatrix}\hat{a}^\epsilon \\ \hat{b}^\epsilon\end{bmatrix}.$$

where

$$\Delta^{(\pm)}(z) = \frac{1}{2}\left(\frac{\rho(z)}{\overline{\rho}} \pm \frac{\overline{K}}{K(z)}\right) = \frac{1}{2}\left(1 \pm 1 \pm \nu\left(\frac{z}{\epsilon^2}\right)\right)$$

Eventually, the propagator equation is

$$\frac{d}{dz}\mathbf{P}_\omega(z) = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\mathbf{P}_\omega(z), \quad \mathbf{P}_\omega(0) = \mathbf{I}_2$$

with

$$\mathbf{H}_\omega(z,\nu) = \frac{i\omega}{2\overline{c}}\nu\begin{bmatrix} 1 & -e^{-2i\omega z/\overline{c}} \\ e^{2i\omega z/\overline{c}} & -1\end{bmatrix}.$$

Another result in asymptotic analysis of random ODEs

Theorem 3 (Averaging in the strongly heterogeneous scaling regime)

Let $X^\epsilon(z)$ for $z \geq 0$ be the process in $\mathbb{R}^d$ defined by the random differential equation

$$\frac{dX^\epsilon}{dz}(z) = \frac{1}{\epsilon} F\left(X^\epsilon(z), Y\left(\frac{z}{\epsilon^2}\right), \frac{z}{\epsilon}\right),$$

starting from $X^\epsilon(0) = x_0 \in \mathbb{R}^d$. Assume that $Y(z)$ is a z-homogeneous Markov ergodic process with the same hypotheses as in Theorem 2. Then the random processes $X^\epsilon(z)$ converge in distribution to the Markov diffusion process $X(z)$ with generator :

$$\mathcal{L}\phi = \frac{1}{Z_0} \int_0^{Z_0} \int_0^\infty \mathbb{E}\left[F(x, Y(0), \tau) \cdot \nabla(F(x, Y(z), \tau) \cdot \nabla\phi)\right] dz d\tau$$

Another result in asymptotic analysis of random ODEs

With the same (lengthy) computations as in the weak-scattering regime, one obtains that, with probability one

$$\lim_{L\to\infty} \frac{1}{L} \ln\left(|\hat{\mathcal{T}}(\omega)|^2\right) = -\frac{1}{L_{\mathrm{loc}}}$$

where the localization length in the strong-scattering regime is the low-frequency limit of the localization length in the weak scattering regime

$$L_{\mathrm{loc}}(\omega) = \frac{4\bar{c}^2}{\gamma(0)\omega^2}.$$

Outline

3D Wave propagation in a randomly-layered medium

The pressure $p(t, z)$ and velocity $\boldsymbol{u}(t, z)$ in a layered medium with density $\rho(z)$ and bulk modulus $K(z)$ verify

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \boldsymbol{0},$$

$$\frac{1}{K_0} \frac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} = 0.$$

We write $\boldsymbol{x} = (x, y)$, and decompose $\boldsymbol{u} = (\boldsymbol{v}, u)$ in the plane orthogonal to $\boldsymbol{e}_z$ and along $\boldsymbol{e}_z$.
We consider Fourier transform in time and space (only horizontal)

$$\hat{p}(\omega, \boldsymbol{\kappa}, z) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i\omega(t - \boldsymbol{\kappa}\cdot\boldsymbol{x})} p(t, \boldsymbol{x}, z) dt d\boldsymbol{x}$$

and similar definition for $\hat{u}(\omega, \boldsymbol{\kappa}, z)$ and $\hat{\boldsymbol{v}}(\omega, \boldsymbol{\kappa}, z)$.

Localization length in a 3D problem with 1D randomness

Eventually, the equation driving $(\hat{p}, \hat{u})$ (for $z \neq z_s$) is

$$\frac{d\hat{p}}{dz} = i\omega\rho(z)\hat{u}$$

$$\frac{d\hat{u}}{dz} = i\omega\left(\frac{1}{K(z)} - \frac{\kappa^2}{\rho(z)}\right)\hat{p}$$

where $\kappa = |\boldsymbol{\kappa}|$. For small enough κ, this equation has the same form as in a 1D medium, with velocity

$$\overline{c}(\kappa) = \frac{\overline{c}}{\sqrt{1 - \kappa^2\overline{c}^2}}$$

and $\overline{c}$ defined as before.
The localization length for a mono-chromatic wave is, for instance in the strong scattering regime, is therefore

$$L_{\mathrm{loc}}(\omega) = \frac{4\overline{c}^2}{\gamma(0)\omega^2\left(1 - \kappa^2\overline{c}^2\right)}.$$

which diverges at $\kappa = 1/\overline{c}$.